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# DISCRETE TIME MODEL PREDICTIVE CONTROL APPROACH FOR INVERTED PENDULUM SYSTEM WITH INPUT CONSTRAINTS

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**Abstract-** Model predictive control (MPC) includes a receding-horizon control techniques based on the process model for predictions of the plant output. Since late 1970's several MPC approaches have been reported in the literature. Selection of the most appropriate MPC approach depend on the specific problem. In this paper, discrete time MPC is applied to a inverted pendulum system coupled to a cart. The objective of the MPC-controller is to drive the system towards pre-calculated trajectories that move the system from one reference point to another. Quadratic programming is used for optimization of objective function (with and without constraints).

**Keywords-** inverted pendulum; model predictive control; quadratic programming; unconstrained and constrained optimization.

## I. INTRODUCTION

Inverted Pendulum is a very good platform for control engineers to verify and apply different logics in the field of control theory. This system is inherently unstable and nonlinear. In the past decades, lots of researches have done with the inverted pendulum equilibrium control and lots of control methods have been tested. Some of them are: linear quadratic regulator and sliding mode control [1], proportional-integral-derivative (PID) control and fuzzy control [2], neural networks [3], etc. For design and control of the inverted pendulum system a mathematical model must first be obtained. From this, the control system can be designed and simulated to obtain optimum control. As keeping the pendulum in the inverted position, it is also desirable to control the horizontal position of the cart on the track. The horizontal position control is required because there is a finite length of the track upon which the cart can move. Mainly, there are two modes of operation for apparatus:

- keeping the pendulum inverted and only ensuring that the cart's position does not move out of range.
- applying a step input to the cart, requesting it to move from one position to another, whilst keeping the pendulum Inverted. The inverted pendulum system has many practical applications like in aircrafts, robotics etc.

In this paper we consider the inverted pendulum coupled to a cart. The purpose of the system is to keep the cart on reference tracking position. This is achieved by designing a model predictive controller which can control the movement of the cart in predefined path when a appropriate force is applied. The MPC-controller is applied to show its applicability to a linearized inverted pendulum system. The paper is organized as follows: The single inverted pendulum model is presented in Section II. Section III describes the formulations of Discrete

time Model Predictive Control used in this work. Section IV discusses formulation of constrained control problem which is used to solve the MPC optimisation problems (Hildreth quadratic programming). Finally, section V describes the simulation results and section VI the conclusions.

## II. INVERTED PENDULUM SYSTEM

The system consists of a cart which can be moved horizontal with the application of a force as shown in Fig. 1.

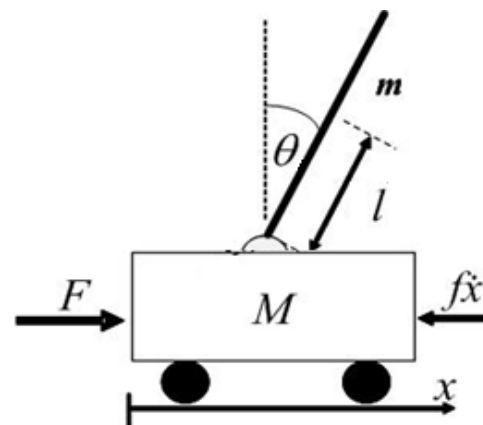


Figure 1. Inverted Pendulum on a cart structure

Forces and moments acting in the system were analysed using Fig. 1 where  $\theta$  represents the angle of pendulum rod,  $M$  and  $m$  stands for the weight of the cart and pendulum respectively,  $l$  is the distance between centre of gravity of the pendulum and the centre of rotation of the pendulum and  $g$  is the gravity acceleration constant. Symbol  $F$  represents the force produced by the DC motor. It is obvious that the position and dynamics of the pendulum affects the cart. This affect is described by a force which can be divided into horizontal and vertical components. The horizontal component of the force is:

$$H = m \frac{\partial^2}{\partial t^2} (x + l \sin \theta) \quad (1)$$

where  $x$  represents position of the cart. The vertical component is:

$$V = m \frac{\partial^2}{\partial t^2} (l \cos \theta) \quad (2)$$

The motion equation of the cart can be written as follows:

$$M \frac{\partial^2 x}{\partial t^2} = F - H - f \frac{\partial x}{\partial t} \quad (3)$$

where  $f$  represents constant of a velocity proportional friction of the cart. According to the angular momentum conservation law, the rotary motion of the rod about its centre is described as:

$$I_s \frac{\partial^2 \theta}{\partial t^2} = Vl \sin \theta - Hl \cos \theta - C \frac{\partial \theta}{\partial t} \quad (4)$$

where  $I_s$  represents the inertia moment of the pendulum rod with respect to the centre of gravity and  $C$  denotes the friction constant of the pendulum[5].

Substituting equations (1) and (2) into equations (3) and (4) leads to the description of behaviour of the system by set of two nonlinear differential equations: [6]

$$M_0 \ddot{x} + f \dot{x} + ml \ddot{\theta} \cos \theta - ml (\dot{\theta})^2 \sin \theta = F \quad (5)$$

$$I \ddot{\theta} + C \dot{\theta} - mlg \sin \theta + ml \ddot{x} \cos \theta = 0 \quad (6)$$

Following substitutions were used in (5) and (6):

$$I = I_s + ml^2$$

$$M_0 = M + m$$

The model described by nonlinear equations (5) and (6) is created in the MATLAB/Simulink environment as a standalone block. The Simulink scheme is shown in Fig. 2.

All constants and symbols are clearly defined in Table I.

#### A. Linearized Model

For the purpose of control design, nonlinear differential equations (5) and (6) describing behavior of inverted pendulum were linearized around some appropriate working point[6]. The state space model matrices  $A$  and  $B$  of the linearized model is directly written as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} \quad (7)$$

Where

$$a_{22} = \frac{If}{R}, a_{23} = \frac{m^2 l^2 g}{R}, a_{24} = \frac{-Cml}{R}$$

$$a_{42} = \frac{-mlf}{R}, a_{43} = \frac{-M_0 mlg}{R}, a_{44} = \frac{CM_0}{R}$$

$$b_2 = \frac{-I}{R}, b_4 = \frac{ml}{R}, R = m^2 l^2 - M_0 I$$

This continuous linear model is discretized with respect to sampling time in design of MPC controller.

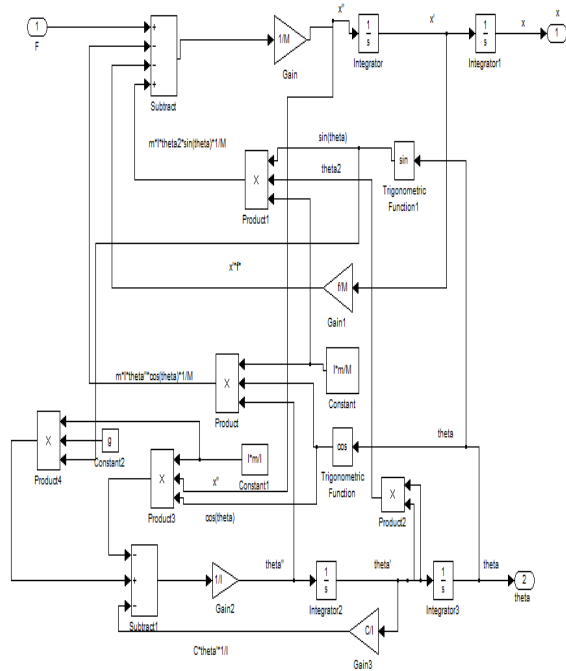


Figure 2. Simulink scheme of non-linear model

TABLE I. parameters of the system

Parameter	Symbol	Value and Unit
Cart weight	$M$	4.0kg
Pendulum weight	$m$	0.36kg
Total weight	$M_0$	4.36kg
Pendulum length	$l$	0.420m
Moment of inertia	$I$	0.08433kg m <sup>2</sup>
Cart friction	$f$	6.5Kg/s
Pendulum friction	$C$	0.00652kg m <sup>2</sup> /s

### III. MODEL PREDICTIVE CONTROL ALGORITHM

Model Predictive Control usually contains the following three ideas.

1. Explicit use of a model to predict the process output along a future time horizon.
2. Calculation of a control sequence to optimize an objective function.

3. A receding horizon strategy, so that at each instant the horizon is moved towards the future, which involves the application of the first control signal of the sequence calculated at each step [8].

The MPC methodology is characterized by the strategy represented in Fig. 2. The inverted pendulum system is considered as single-input single-output (SISO) system. The input of the system is force on the cart (F) and output of the system is cart position (r). Control objective is to move the cart according to the predefined reference trajectory for unconstrained and constrained optimization. Here pendulum angle is taken as a constrained because it should be close to zero during whole control process.

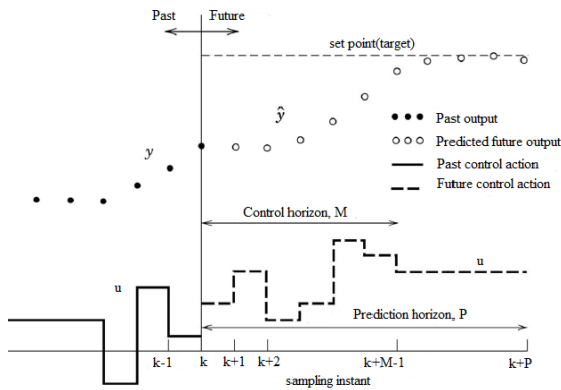


Figure 3. Control strategy of Model Predictive Control

#### A. Model structure

The state space model of the linearized system is described as;

$$x_m(k+1) = A_m x_m(k) + B_m u(k) \quad (8)$$

$$y(k) = C_m x_m(k) + D_m u(k)$$

However, due to the principle of receding horizon control, where a current information of the plant is required for prediction and control, we have implicitly assumed that the input  $u(k)$  cannot affect the output  $y(k)$  at the same time. Thus,  $D_m = 0$  in the plant model.

$$y(k) = C_m x_m(k) \quad (9)$$

by augmenting the state variable  $x_m(k)$  to include the output variable  $y(k)$  we have the augmented state space model [9].

$$\begin{bmatrix} \Delta x_m(k+1) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} A_m & 0_m^T \\ C_m A_m & 1 \end{bmatrix} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix} \Delta u(k) \quad (10)$$

$$y(k) = \begin{bmatrix} 0_m & 1 \end{bmatrix} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} \quad (11)$$

where  $0_m = [0 \ 0 \ \dots \ 0]$ .

#### B. Prediction of state and output variables

Assuming that at the sampling instant  $k_i$ ,  $k_i > 0$ , the state variable vector  $x(k_i)$  is available through measurement, the state  $x(k_i)$  provides the current plant information. Following the standard approach in MPC, the future control trajectory is denoted by:

$$\Delta U = [\Delta u(k_i), \Delta u(k_i + 1), \dots, \Delta u(k_i + N_c - 1)]^T \quad (12)$$

The future state variables are denoted as:

$$X = [\Delta x(k_i + 1/k_i), \Delta x(k_i + 2/k_i), \dots, \Delta x(k_i + N_p/k_i)]^T \quad (13)$$

Where  $N_c$  is the control horizon dictating no. of parameters used to capture the future control trajectory and  $N_p$  is the prediction horizon. The control horizon is always less than the prediction horizon.

The predicted output variables are:

$$Y = [y(k_i + 1/k_i), y(k_i + 2/k_i), \dots, y(k_i + N_p/k_i)]^T \quad (14)$$

The predicted output variable  $Y$  is related to future control trajectory and current state measurement  $x(k_i)$  via the following equation:

$$Y = Fx(k_i) + \phi \Delta U \quad (15)$$

Where

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}, \phi = \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2 B & CAB & CB & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ CA^{N_p-1} B & CA^{N_p-2} B & \dots & \dots & CA^{N_p-N_c} B \end{bmatrix}$$

#### C. Optimization

For a given set-point signal  $r(k_i)$  at sample time  $k_i$ , within a prediction horizon the objective of the predictive control system is to bring the predicted output as close as possible to the set-point signal, where we assume that the setpoint signal remains constant in the optimization window.

Assuming that the data vector that contains the set-point information is

$$R_s^T = [1 \ 1 \ 1 \ \dots \ 1] r(k_i)$$

Objective function  $J$  is defined as

$$J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U \quad (16)$$

where the first term is error between the predicted output and the set-point signal while the second term is the consideration given to the size of  $\Delta$  when the objective function  $J$  is made to be as small as possible.  $\bar{R}$  is a diagonal matrix in the form that

$$\bar{R} = r_w I_{N_c \times N_c}$$

where  $r_w$  is used as a tuning parameter for the desired closed-loop performance.

The necessary condition for minimizing  $J$  is

$$\frac{\partial J}{\partial \Delta U} = 0,$$

From which the optimal solution of control signal is find as

$$\Delta U = (\phi^T \phi + \bar{R})^{-1} \phi^T (R_s - Fx(k_i)) \quad (17)$$

The matrix  $(\phi^T \phi + \bar{R})^{-1}$  is called Hessian matrix.

#### IV. DISCRETE TIME MPC WITH CONSTRAINTS

In this section solution to the constrained control problem is discussed. In the case of constraints in manipulated variable, we express:

$$\begin{bmatrix} u(k_i) \\ u(k_i+1) \\ u(k_i+2) \\ \vdots \\ u(k_i+N_c-1) \end{bmatrix} = \begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix} u(k_i-1) + \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & 0 & 0 \\ I & I & I & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I & I & I & I & \cdot & I \end{bmatrix} \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix} \quad (18)$$

Writing Eq. (18) in a compact matrix form, with  $C_1$  and  $C_2$  corresponding to the appropriate matrices, then the constraints for the control movement are imposed as

$$-(C_1 u(k_i - 1) + C_2 \Delta u) \leq -U_{\min} \quad (19)$$

$$(C_1 u(k_i - 1) + C_2 \Delta u) \leq U_{\max} \quad (20)$$

Where  $U_{\min}$  and  $U_{\max}$  are column vectors with  $N_c$  elements of  $U_{\min}$  and  $U_{\max}$  respectively. Similarly, the constraints on increment of control signal is defined as

$$-\Delta U \leq -\Delta U_{\min} \quad (21)$$

$$\Delta U \leq \Delta U_{\max} \quad (22)$$

Where  $\Delta U_{\min}$  and  $\Delta U_{\max}$  are column vectors with  $N_c$  elements of  $\Delta U_{\min}$  and  $\Delta U_{\max}$  respectively. The output constraints are defined as

$$Y_{\min} \leq Fx(k_i) + \phi \Delta U(k) \leq Y_{\max} \quad (23)$$

The objective function that has to minimize is

$$J = (R_s - Fx(k_i))^T (R_s - Fx(k_i)) - 2\Delta U^T \phi^T (R_s - Fx(k_i)) + \Delta U^T (\phi^T \phi + \bar{R}) \Delta U \quad (24)$$

Subject to inequality constraints

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \Delta U \leq \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, \quad (25)$$

$$\text{Where } M_1 = \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix}, M_2 = \begin{bmatrix} -I \\ I \end{bmatrix}, M_3 = \begin{bmatrix} -\phi \\ \phi \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -U_{\min} + C_1 u(k_i - 1) \\ U_{\max} - C_1 u(k_i - 1) \end{bmatrix}, N_2 = \begin{bmatrix} -\Delta U_{\min} \\ -\Delta U_{\max} \end{bmatrix}, N_3 = \begin{bmatrix} -Y_{\min} + Fx(k_i) \\ Y_{\max} - Fx(k_i) \end{bmatrix}$$

#### A. Numerical Solutions Using Hildreth Quadratic Programming

The objective function J and the constraints are expressed as

$$J = 0.5 \Delta U^T M \Delta U + \Delta U^T g \quad (26)$$

$$\text{Subject to } A_c \Delta U \leq B_c \quad (27)$$

Where M is the positive definite symmetric matrix. The necessary conditions for this optimization problem (Kuhn-Tucker conditions) are

$$M \Delta U + g + A_c^T \lambda = 0$$

$$A_c \Delta U - B_c \leq 0$$

$$\lambda^T (A_c \Delta U - B_c) = 0$$

$$\lambda \geq 0 \quad (28)$$

where the vector  $\lambda$  contains the Lagrange multipliers. Let  $S_{act}$  denote the index set of active constraints.

Then the necessary conditions become

$$M \Delta U + g + \sum_{i \in S_{act}} \lambda_i A_{ci}^T = 0$$

$$A_{ci} \Delta U - B_{ci} = 0 \quad i \in S_{act}$$

$$A_{ci} \Delta U - B_{ci} < 0 \quad i \notin S_{act}$$

$$\lambda_i \geq 0 \quad i \in S_{act}$$

$$\lambda_i = 0 \quad i \notin S_{act} \quad (29)$$

Where  $A_{ci}$  is the  $i$ th row of the  $A_c$  matrix. Suppose an active set is guessed and the corresponding equality constrained problem is solved. Then if the other constraints are satisfied and the Lagrange multipliers turn out to be nonnegative, that solution would be correct.

A dual method can be used to identify the constraints that are not active. Not only can they be eliminated in the solution, but also they can offer additional insights into the constrained control problem. The dual problem to the original quadratic problem [10] is described as follows. Assuming feasibility, the problem is equivalent to

$$\max_{\lambda \geq 0} \min_{\Delta U} [0.5 \Delta U^T M \Delta U + \Delta U^T g + \lambda^T (A_c \Delta U - B_c)] \quad (30)$$

The minimization over  $\Delta U$  is unconstrained and is attained by;

$$\Delta U = -M^{-1} (g + A_c^T \lambda) \quad (31)$$

Substituting (31) in (30), the dual problem becomes

$$\max_{\lambda \geq 0} (-0.5 \lambda^T H \lambda - \lambda^T K - 0.5 g^T M^{-1} g) \quad (32)$$

Where  $H = A_c M^{-1} A_c^T$  and  $K = B_c + A_c M^{-1} g$ .

Equation (32) is equivalent to

$$\min_{\lambda \geq 0} (0.5 \lambda^T H \lambda + \lambda^T K + 0.5 B_c^T M^{-1} B_c) \quad (33)$$

$\lambda_i$  is adjusted to minimize objective function. If  $\lambda_i < 0$  is required set  $\lambda_i = 0$ . In any case, the objective function is decreased. Then we consider the next component  $\lambda_{i+1}$ .

$\lambda_i^{m+1} = \max(0, w_i^{m+1})$  where

$$w_i^{m+1} = -\frac{1}{h_{ii}} \left[ K_i + \sum_{j=1}^{i-1} h_{ij} \lambda_j^{m+1} + \sum_{j=i+1}^{nn} h_{ij} \lambda_j^m \right] \quad (34)$$

nn is the dimension of  $\lambda$ .

The optimal solution to the constrained control problem (29) is given by the solution of the linear equations

$$\begin{bmatrix} M & A_{c_{act}}^T \\ A_{c_{act}} & 0 \end{bmatrix} \begin{bmatrix} \Delta U \\ \lambda_{act} \end{bmatrix} = \begin{bmatrix} -g \\ B_{c_{act}} \end{bmatrix} \quad (35)$$

$$\lambda_{act} = -(A_{c_{act}} M^{-1} A_{c_{act}}^T)(B_{c_{act}} + A_{c_{act}} M^{-1} g) \quad (36)$$

$$\Delta U = -M^{-1}(g + A_{c_{act}}^T \lambda_{act}) \quad (37)$$

## V. SIMULATION RESULTS

This section shows the results obtained for unconstrained and constrained optimization problem for inverted pendulum system. The prediction horizon  $N_p$  is 20 and control horizon  $N_c$  is 3. The control input is constrained via  $-10 \leq \Delta U \leq 10$ . Hildreth quadratic programming is used to solve the optimization problem. Sampling time is 0.1s. Fig.4 shows that the cart position tracks the reference and pendulum angle remains in upright position for unconstrained optimization problem and Fig.5 shows that the objective function is minimized at every time instant.. Using Hildreth quadratic programming algorithm the constrained problem is solved and the results are shown in Fig.6 and Fig.7.

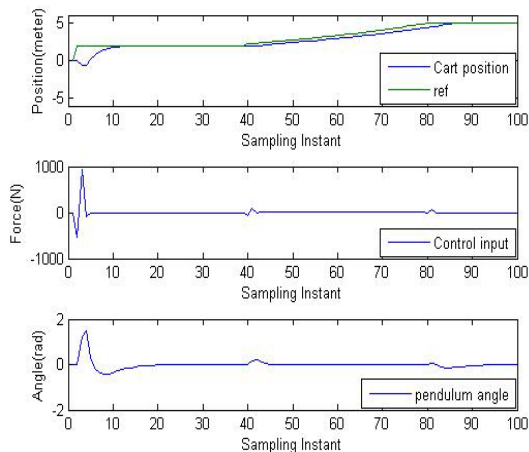


Figure 4. Unconstrained Optimization( $N_p=20, N_c=3$ )

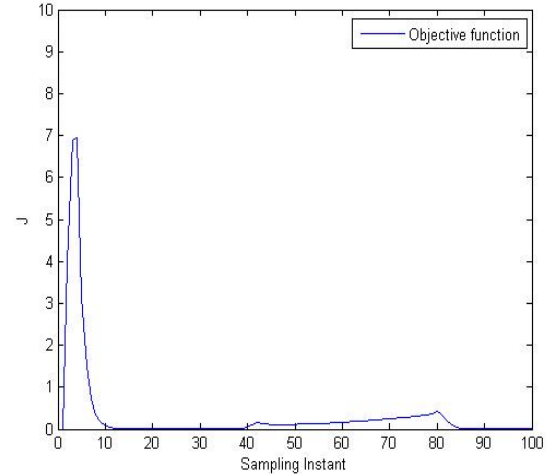


Figure 5. Optimal Cost

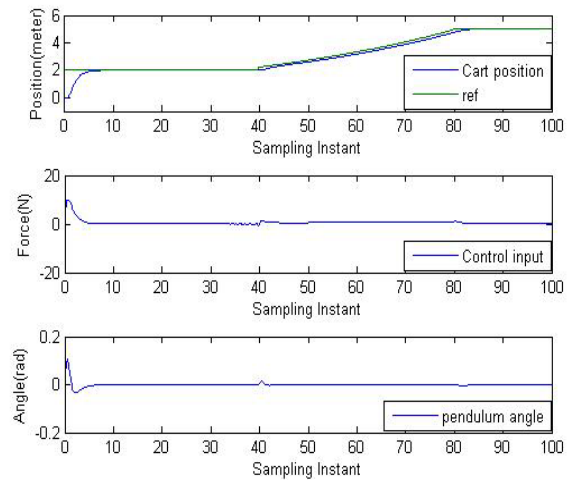


Figure 6. Constrained Optimization( $N_p=20, N_c=3$ )

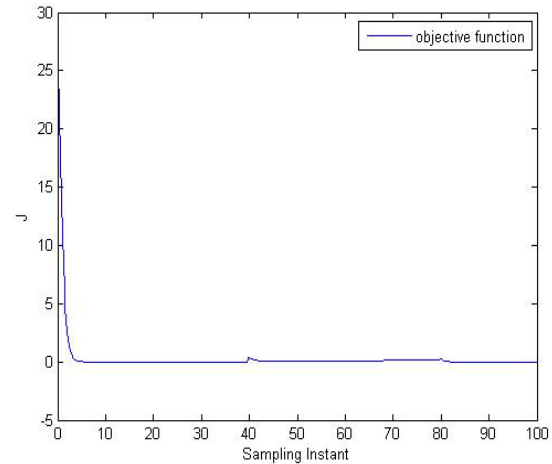


Figure 7. Optimal Cost

## VI. CONCLUSION

In this paper a discrete time model predictive controller is designed for controlling of inverted pendulum system. We see that cart position tracks the predicted reference path for both the constrained and unconstrained problem. By applying Hildreth Algorithm, the constrained optimization problem is

solved. The control sequence obtained is optimal, the pendulum is in inverted position and the objective function is minimized. Future works could extend this investigation by considering other control techniques (not restricted to MPC), as well as the use of different objective functions.

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