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A FRACTIONAL DELAY FIR FILTER BASED ON LAGRANGE INTERPOLATION OF FARROW STRUCTURE

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Abstract - An efficient implementation technique for the Lagrange interpolation is derived. This formulation called the Farrow structure leads to a version of Lagrange interpolation that is well suited to time varying FD filtering. Lagrange interpolation is mostly used for fractional delay approximation as it can be used for increasing the sampling rate of signals and systems. Lagrange interpolation is one of the representatives for a class of polynomial interpolation techniques. The computational cost of this structure is reduced as the number of multiplications are minimised in the new structure when compared with the conventional structure.

***Keywords** - Farrow structure, Lagrange interpolation, Horner's method, Fractional delay (FD), Finite impulse response (FIR) filter.*

I. INTRODUCTION

A fractional delay filter is a filter of digital type having the main function so as to delay the processed input signal as a fractional of the sampling period time. There are several applications where such fractional signal delay value is required, examples of such systems are: timing adjustment in all-digital receivers (symbol synchronization), conversion between arbitrary sampling frequencies, echo cancellation, speech coding and synthesis, musical instruments modelling etc.

In order to achieve the fractional delay filter function, two main frequency-domain specifications are required by the filter. The filter magnitude frequency response must have an all-pass behaviour in a wide frequency range, as well as it is necessary to have a phase frequency response that must be linear with a fixed fractional slope through the bandwidth.

There are two main design approaches: time-domain and frequency-domain design methods. In first one, the fractional delay filter coefficients are easily obtained through classical mathematical interpolation formulas, but there is a small flexibility to meet frequency-domain specifications. On the other hand, the frequency-domain methods are based on frequency optimization process, and a more frequency specification control is available. One important result of frequency-domain design methods is a highly efficient implementation structure called Farrow structure which allows online fractional value update.

Farrow structure and the modified Farrow structure are two very efficient approaches for adjustable fractional delay filtering, which allows online fractional delay value update with a fixed set of parallel FIR branch filters and only one control parameter. Both structures are composed of L+1 branch FIR filters $C_l(z)$, each one with length N.[3]

II. IDEAL FRACTIONAL DELAY

The continuous-time output signal $y(t)$ of ageneral signal delay system is defined by:

$$y(t) = x(t-t_l) \qquad (1)$$

Where x(t) is the continuous-time input signal and $t_l$the obtained time delay value.

In a discrete-time system, the input-output relationship of a signal delay system is expressed as:

$$y(lT) = x(nT-DT) \qquad (2)$$

Where the delay value is given by DT, $y(lT)$ and $x(nT)$ are the discrete-time versions of output and input signals, respectively, and T is the sampling period time.

A signal delay value equal to a multiple of the sampling period, D as an integer N, can be easily implemented in a discrete-time system using the signal value for a time of NT:

$$Y(lT) = x(nT-NT) \qquad (3)$$

Here the signal delay value is limited to be only N time the sampling period.

The simplified block diagram for a fractional delay filter is shown in fig.1, which output for a non causal FD FIR filter is given by the discrete-time convolution:

$$y(lT) = \sum_{k=N/2}^{\frac{N}{2}-1} x(n-k)h(k,\mu) \qquad (4)$$

Where N is the length of the FD filter. The system function H(z) of the FD filter can be expressed as:

$$H(z) = z^{-D} \qquad (5)$$

Where the delay value is given as: $D = D_{fix} + \mu_l$, $D_{fix}$is a fixed delay value andμ_l is the desired fractional delay value.

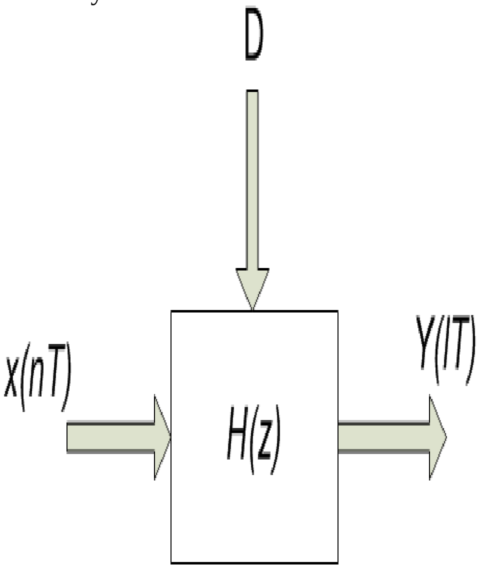

Fig.1 : Simplified block diagram for a FD filter

III. LAGRANGE INTERPOLATION

The Lagrange interpolation method is based on the well-known result in polynomial algebra that using an *N*th-order polynomial it is possible to match N+1 given arbitrary points.

Lagrange interpolation has two favourable features:

1. it is accurate at low frequencies and
2. it never overestimates the amplitude of the signal when the delay has been chosen so that $(N-1)/2 \leq D \leq (N+1)/2$.

The first advantage is justified by the fact that most of the signals are lowpass signals and thus it is advisable to use an approximation technique that has the smallest error at low frequencies.

The second property is called passivity and it implies that the magnitude response of the Lagrange interpolator is less than or equal to one for the mentioned values of D. Always the magnitude response of the Lagrange interpolator exceeds unity when the delay parameter is out of the optimal range.

The use of the Lagrange interpolation method has three main advantages:

1) The easeto compute the FDF coefficients from one closed form equation,
2) The FDF magnitudefrequency response at low frequencies is completely flat,
3) A FDF with polynomial-definedcoefficients allows the use of an efficient implementation structure called Farrow structure.

The magnitude and phase responses of the third-order Lagrange interpolators are shown in fig. 2(a), 2(b).

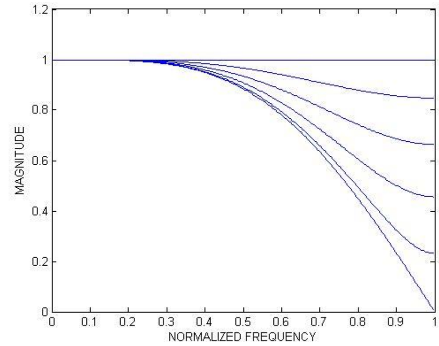

Fig. 2(a) : Magnitude response of Lagrange interpolation

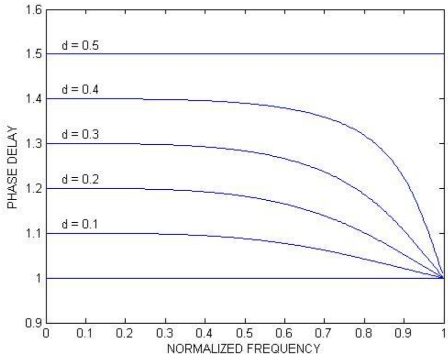

Fig. 2(b) : Phase response of Lagrange interpolation

IV. FARROW STRUCTURE OF LAGRANGE INTERPOLATION

Lagrange interpolation is usually implemented using a direct-form FIR filter structure. An alternative structure is obtained approximating the continuos-time function $x_c(t)$ by a polynomial in D, which is the interpolation interval or fractional delay. The interpolants, i.e., the new samples, are now represented by

$$y(n) = x(n-D) = \sum_{k=0}^{N} c(k)D^k \quad (6)$$

That takes on the value $x(n)$ when $D = n$. The coefficients $c(k)$ are solved from a set of N+1 linear equations. Farrow has suggested that every filter coefficient of an FIRinterpolating filter could be expressed as an *N*th-order polynomial in the delay parameter*D*. Hence this result in *N*+1 FIR filters with constant coefficients.

The alternative implementation for Lagrange interpolation is obtained formulatingthe polynomial interpolation problem in the z-domain as

$$Y(z) = H(z)X(z) \quad (7)$$

Where X(z) and Y(z) are the z-transforms of the input and output signal, x(n) and y(n),respectively, and the transfer function H(z) is now expressed as a polynomial in D.

$$H(z) = \sum_{k=0}^{N} C_k(z)\, D^k \quad (8)$$

The familiar requirement that the output sample should be one of the input samples forinteger D may be written in the z-domain as

$Y(z) = z^{-D}X(z)$ for D = 0,1,2,K,N (9)

Together with Eqs. (8) and (9) this leads to the following N + 1 conditions

$$\sum_{k=0}^{N} C_k(z)\, D^k = z^{-D}\ for\ D = 0,1,2 \dots N \tag{10}$$

This may be expressed in matrix form as

$$Uc = z \tag{11}$$

Where the L×L matrix U is given by

$$U = \begin{bmatrix} 0^0 0^1 & 0^2 & & 0^N \\ 1^0 1^1 & 1^2 & \cdots & 1^N \\ 2^0 2^1 & 2^2 & & 2^N \\ \vdots & & \ddots & \vdots \\ N^0 N^1 & N^2 & \cdots & N^N \end{bmatrix} = \begin{bmatrix} 10 & 0 & & 0 \\ 11 & 1 & \cdots & 1 \\ 12 & 4 & & 2^N \\ \vdots & & \ddots & \vdots \\ 1\,N & N^2 & \cdots & N^N \end{bmatrix} \tag{12}$$

vector c is

$$c = [C_0(z) C_1(z) C_2(z)\ \dots\ C_N(z)]^T \tag{13}$$

and the delay vector

$$z = [1\ \ z^{-1} z^{-2}\ \dots\ z^{-N}]^T \tag{14}$$

The matrix **U** has the Vandermonde structure and thus it has an inverse matrix $\mathbf{U}^{-1}$.

The solution of Eq. can thus be written as

$$c = U^{-1}z \tag{15}$$

The inverse matrix $\mathbf{U}^{-1}$, that we shall denote by **Q**, may be solved using Cramer's rule.

The rows of the inverse Vandermonde matrix **Q** contain the filter coefficients used inthe new structure, and thus it is convenient to write

$$Q = [q_0 q_1\, q_2\ \dots\ q_N]^T \tag{16}$$

The transfer functions *Cn*(*z*) are thus obtained by inner product as

$$C_n(z) = q_n z = \textstyle\sum_{k=0}^{N} q_n(k)\, z^{-k}\ for\ n = 1,2, \dots N \tag{17}$$

The coefficients *qn*(*k*) for the FIR filters *Cn*(*z*) are computed inverting the Vandermondematrix **U**.

By setting *D* = 0 in Eq. (10), it is seen that

$$\textstyle\sum_{k=0}^{N} C_k(z)\, 0^k = 1\ \Rightarrow C_0 \equiv 1 \tag{18}$$

This implies that the transfer function $C_0(z) = 1$ regardless of the order of the interpolator.

The other transfer functions *Cn*(*z*) given by Eq. (17) are *N*th-order polynomials in*z*-1 , that is, they are *N*th-order FIR filters. The interpolator is directly controlled by the fractional delay D, i.e., no computationally intensive coefficient update is needed when D is changed.

Farrow structure is most efficiently implemented using Horner's method that is

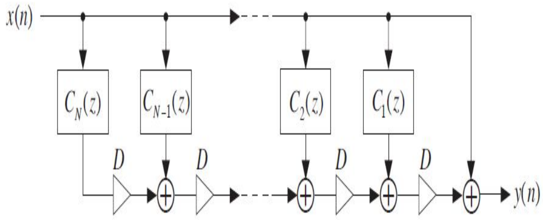

Fig. 3 : Farrow structure of Lagrange interpolation implemented using Horner's method.

$$\sum_{k=0}^{N} C_k(z)\, D^k = C_0(z) + [C_1(z) + [C_2(z) + \cdots + [C_{N-1}(z) + C_N(z)D]D \dots]D \tag{19}$$

In this method *N* multiplications by *D* are needed. A general *N*th-order Lagrangeinterpolator that employs the suggested approach is shown in Fig. 3. Since there isno need for the updating of coefficients, this structure is particularly well suited toapplications where the fractional delay *D* is changed often, even after every sampleinterval.

If the delay is constant or updated very seldom, it is recommended to implementLagrange interpolation using the standard FIR filter structure because it is computationallyless expensive. Namely, with the FIR filter structure *N* + 1 multiplications and*N* additions are needed. In Farrow's structure, there are *N* pieces of *N*th-order FIR filterswhich results in *N*(*N* +1) multiplications and N^2 additions. There are also *N* multiplicationsby *D* and *N* additions. Altogether this means $N^2 + 2N$ multiplications and$N^2 + N$ additions per output sample.

As examples, let us solve for the transfer functions $C_n(z)$ for linear interpolation andsecond-order Lagrange interpolation. For *N* = 1, Eq. (11) yields

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_0(z) \\ C_1(z) \end{bmatrix} = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \tag{20}$$

The solution is given by

$$\begin{bmatrix} C_0(z) \\ C_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 + z^{-1} \end{bmatrix} \tag{21}$$

It is seen that $C_1(z) = z^{-1}$-1 when *N* = 1. The overall transfer function *H*(*z*) of theFarrow structure of linear interpolation is written as

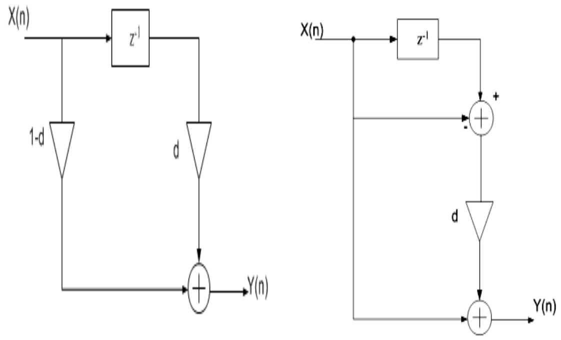

Fig. 4(a) : The direct-form FIR filer structure for linear interpolation and Fig. 4(b) the equivalent Farrow structure

$$H(z) = 1 + (z^{-1} - 1)D \quad (22)$$

Note that in this case $D = d$. The linear interpolator may thus be implemented by the structure illustrated in Fig. 4b. It is seen that the Farrow structure is as efficient as the direct-form non-recursive structure (Fig. 4a) when $N = 1$, since the number of operations is the same in both. If multiplication is more expensive than addition, like it is in VLSI implementations, then the Farrow structure (Fig. 4b) is preferable.

For N = 2 eq. is written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} C_0(z) \\ C_1(z) \\ C_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix} \quad (23)$$

Now the inverse vandermonde matrix Q is given by

$$Q = U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \quad (24)$$

The transfer functions thus obtained are

$$C_0(z) = 1, C_1(z) = -\frac{3}{2} + 2z^{-1} - \frac{1}{2}z^{-2},\ C_2(z) = \frac{1}{2} - z^{-1} + \frac{1}{2}z^{-2} \quad (25)$$

And the overall transfer function H(z) can be written as

$$H(z) = C_0(z) + C_1(z)D + C_2(z)D^2 \quad (26)$$

Farrow form of parabolic or second-order Lagrange interpolation is illustrated in Fig. 5. In this example it is seen that the transfer functions $C_n(z)$ can share unit delays, because they all use the same delayed signal values $x(n - k)$ with $k = 0, 1, 2, ..., N$. Furthermore, the number of multiplications can be reduced in a practical implementation.It may be taken into account that some of the coefficients have the value 1 or -1thus eliminating the need for a multiplication and that the corresponding coefficients of two transfer functions can be equal.

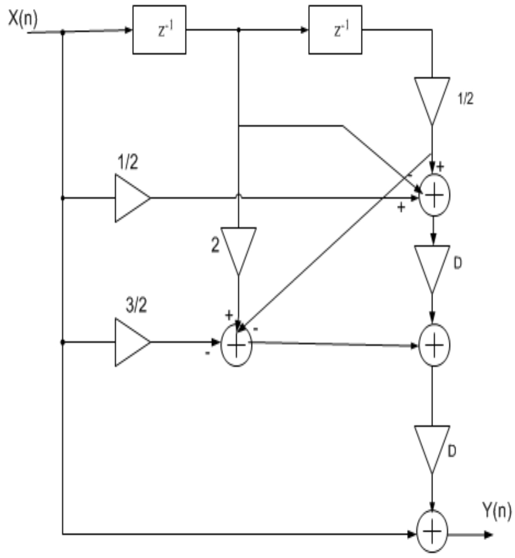

Fig. 5 : Farrow structure for a second order Lagrange interpolator

Modified Farrow structure

Farrow structure can be made more efficient changing the range of the parameter *D*so that the integer part is removed. The new parameter range is 0 ≤*d* ≤1 (for odd *N*) or-0.5 ≤*d* ≤0.5 (for even *N*). This change can be obtained introducing a transformationmatrix **T** defined by

$$T_{n,m} = \begin{cases} round\left(\frac{N}{2}\right)^{n-m} \binom{n}{m} & for\ n \geq m \\ 0\ for\ n < m \end{cases} \quad (27)$$

Where n, m = 0, 1, 2, ..., N

For N = 2 this matrix is expressed as

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad (28)$$

Multiplying the coefficient matrix **Q** by matrix (3.110) a modified coefficient matrix is obtained as

$$\boldsymbol{Q} = \boldsymbol{TQ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \quad (29)$$

This transformation is equivalent to substituting $D' = D + 1$. The FIR filters of themodified structure are written as

$$\mathfrak{C}_0(z) = z^{-1}, \mathfrak{C}_1(z) = -\frac{1}{2} + \frac{1}{2}z^{-2}, \mathfrak{C}_2(z) = \frac{1}{2} - z^{-1} + \frac{1}{2}z^{-2} \quad (30)$$

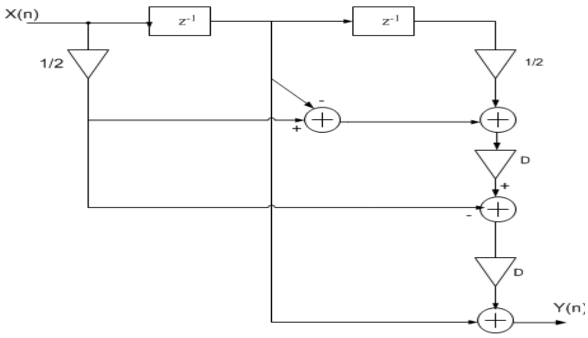

Fig. 6 : Modified Farrow structure for a second order Lagrange interpolator.

V. RESULTS

Table 1 represents the arithmetic complexity for the various orders of the Farrow structure of Lagrange interpolation. The usage of multipliers and adders are tabulated below as follows:

TABLE I

Order of the Farrow structure	**Adders**	**Multipliers**
Second order	6	6
Modified second order	4	4

Table II and III represents the optimization results for various FD FIR filters.

TABLE II

Order of the Farrow structure	**Area (μm^2)**	**Cell count**	**Power (µw)**	**Delay (ns)**
First order	4264	358	1.63	2.47
Second order	10100	931	3.55	3.32
Modified second order	9999	900	3.03	3.24

TABLE III

FD FIR filter	**Area (μm^2)**	**Power (µw)**	**Delay (ns)**
Horner's method	3568	1.13	5.367
Modified Second order Farrow	4832	0.78	0.323

VI. CONCLUSIONS

In this paper an efficient technique for the FD FIR filter has been proposed using the Lagrange interpolation of the Farrow structure. Lagrange interpolation is preferable for the applications where a low order FD filter is needed. A new implementation structure of the Farrow structure based Lagrange interpolation is derived. Farrow structure of Lagrange interpolation is being compared with the modified Farrow structure of Lagrange interpolation and it is noticed that the complexity, power and delay is optimized.

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